

Orbifold regularity of weak Kähler-Einstein metrics

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1 Introduction

In the resolution of the YTD conjecture on the existence of Kähler-Einstein metrics on Fano manifolds (see [23] and also [5]), a crucial tool is a compactness result. In its simplest form, this result says that the Gromov-Hausdorff limit of a sequence of smooth Kähler-Einstein manifolds $(X_i, \omega_{i, \text{KE}})$ is a normal Fano variety $X := X_\infty$ with klt singularities and that there is a weak Kähler-Einstein metric $\omega_{\infty, \text{KE}}^w$ on X_∞ . The existence of a Gromov-Hausdorff limit follows from Gromov's compactness theorem. So the important information in this statement is about the regularity of X_∞ . It was the second author ([20], [22], see also [15]) who first pointed out the route to prove that X_∞ is an algebraic variety is to establish a so-called partial C^0 -estimate. He demonstrated in [20] how to achieve this when the complex dimension n is equal to 2 by showing that a sequence of Kähler-Einstein surfaces converges to a Fano orbifold with a smooth orbifold Kähler-Einstein metric. Note that when $n = 2$, klt singularities are nothing but quotient singularities or orbifold singularities. Two key ingredients to prove the partial C^0 -estimate in dimension 2 are orbifold compactness result of Einstein 4-manifolds and Hörmander's L^2 -estimates.

Recently, Donaldson-Sun [7] and the second author [21] generalized the partial C^0 -estimate to higher dimensional Kähler-Einstein manifolds. Here they need to rely on compactness results of higher dimensional Einstein manifolds developed by Cheeger-Colding and Cheeger-Colding-Tian (see [4] and the reference therein). Compared to the complex dimension 2 case, the second author also conjectured that $\omega_{\infty, \text{KE}}$ is a smooth orbifold metric away from analytic subvarieties of complex codimension 3. Note that in [4], it was proved that the (metric) singular set of X_∞ has complex codimension at least 2.

It can be shown that, by partial C^0 -estimate, there is a uniform C^2 -estimate of the potential of $\omega_{\infty, \text{KE}}^w$ on X_∞^{reg} . Then the Evans-Krylov theory or Calabi's 3rd derivative estimate allows one to show that $\omega_{\infty, \text{KE}}^w$ is smooth on X_∞^{reg} (see [20], [7], [23]). Alternatively using Păun's Laplacian estimate in [16] and Evans-Krylov theory, Berman-Boucksom-Eyssidieux-Guedj-Zeriahi [1] showed directly that any weak Kähler-Einstein metric ω_{KE}^w on a klt Fano variety X_∞ is smooth on X_∞^{reg} . The purpose of this note is to answer the question by the second author about the regularity of ω_{KE}^w on the orbifold locus X_∞^{orb} of X_∞ . First, if $(X, -K_X)$ is a klt Fano variety, then by [8, Proposition 9.3] there exists a closed subset $Z \subset X$ with $\text{codim}_X Z \geq 3$ such that $X \setminus Z$ has quotient singularities. So we just need to show the following regularity result. For the definition of weak Kähler-Einstein metric, see Definition 1.

Theorem 1. *Assume that ω_{KE}^w is a weak Kähler-Einstein metric on X_∞ . Then ω_{KE}^w is a smooth orbifold metric on X_∞^{orb} .*

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Our proof now uses the existence of an orbifold resolution, i.e., Theorem 3 which is proved by algebraic method. However, we believe that it is not necessary. There should be a purely differential geometric proof of Theorem 1 which does not rely on Theorem 3. In a subsequent paper, we will analyze further structures of singularities of higher codimension. We believe that our analysis can be used to yield a complete understanding of the singularity for any 3-dimensional weak Kähler-Einstein metrics.

2 Regularity on the orbifold locus

From now on we will denote by X any \mathbb{Q} -Fano variety with klt singularities. Assume $\iota : X \rightarrow \mathbb{P}^N$ is an embedding given by the linear system $|-mK_X|$ for $m > 0 \in \mathbb{Z}$ sufficiently large and divisible. Let $h_0 = (\iota^* h_{FS})^{1/m}$ be the pull back of the Fubini-Study Hermitian metric h_{FS} on $\mathcal{O}_{\mathbb{P}^N}(1)$ normalized to be a Hermitian metric on $-K_X$. The Chern curvature form of h_0 is

$$\omega_0 = -\sqrt{-1}\partial\bar{\partial}\log h_0$$

which is a positive $(1,1)$ -current on X . ω_0 is a smooth positive definite $(1,1)$ -form on X^{reg} . However, on the singular locus X^{sing} , ω_0 in general is not canonically related to the local structure of X . Assume $p \in X^{\text{orb}}$ is a quotient singularity. By this, we mean that there exists a small neighborhood \mathcal{U}_p which is isomorphic to a quotient of a smooth manifold by a finite group. In other words, there exists a branched covering map $\tilde{\mathcal{U}}_p \rightarrow \tilde{\mathcal{U}}_p/G \cong \mathcal{U}_p$. The lifting of metric ω_0 to the cover $\tilde{\mathcal{U}}_p$ in general is degenerate.

Now we define an adapted volume form on X by

$$\Omega = |v^*|_{h_0}^{2/m} (v \wedge \bar{v})^{1/m}.$$

Here v is any local generator of $\mathcal{O}(mK_X)$ and v^* is the dual generator of $\mathcal{O}(-mK_X)$. The Kähler-Einstein equation

$$\text{Ric}(\omega_\phi) = \omega_\phi. \quad (1)$$

can be transformed into a complex Monge-Ampère equation:

$$(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n = e^{-\phi}\Omega. \quad (2)$$

Definition 1. A weak solution to the (2) is a bounded function $\phi \in L^\infty(X) \cap \text{PSH}(X, \omega)$ satisfying (2) in the sense of pluripotential theory.

Let's first recall the method to prove the regularity of ϕ on X^{reg} following [1]. One first chooses a resolution $\pi : \tilde{X} \rightarrow X$ with simple normal crossing exceptional divisor $E = \pi^{-1}(X^{\text{sing}})$ such that π is an isomorphism over X^{reg} . Then we can pull back the equation (2) to \tilde{X} and get:

$$(\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-\psi}\pi^*\Omega. \quad (3)$$

On the other hand we can write:

$$K_{\tilde{X}} = \pi^*K_X + \sum_{i=1}^r a_i E_i - \sum_{j=1}^s b_j F_j,$$

such that $E = \cup_{i=1}^r E_i \cup \cup_{j=1}^s F_j$ and $a_i > 0$, $b_j > 0$. The klt property implies: $a_i > 0$, and $0 < b_j < 1$. Analytically, choosing a smooth Kähler metric η on \tilde{X} , there exists $f \in C^\infty(\tilde{X})$ such that:

$$\pi^*\Omega = e^f \frac{\prod_{i=1}^r |s_i|^{2a_i}}{\prod_{j=1}^s |\sigma_j|^{2b_j}} \eta^n.$$

where s_i and σ_j are defining sections of E_i for F_j respectively and $|s_i|^2$ and $|\sigma_j|^2$ are some fixed hermitian norms of them. So we have:

$$(\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\psi)^n = e^{-\psi+f+\sum_i a_i \log |s_i|^2 - \sum_j b_j \log |\sigma_j|^2} \eta^n = e^{\psi_+ - \psi_-} \eta^n, \quad (4)$$

Here we have denoted

$$\psi_+ = f + \sum_i a_i \log |s_i|^2, \quad \psi_- = \psi + \sum_j b_j \log |\sigma_j|^2.$$

It's easy to see that they satisfy the quasi-plurisubharmonic condition:

$$\sqrt{-1}\partial\bar{\partial}\psi_+ \geq -C\eta, \quad \sqrt{-1}\partial\bar{\partial}\psi_- \geq -C\eta, \quad (5)$$

for some uniform constant $C > 0$. To get Laplacian estimate of ψ away from $Z = \pi^{-1}(X_\infty)$, we can first regularize (4) to

$$(\omega_\epsilon + \sqrt{-1}\partial\bar{\partial}\psi_\epsilon)^n = e^{\psi_{+, \epsilon} - \psi_{-, \epsilon}} \eta^n. \quad (6)$$

where $\omega_\epsilon = \pi^*\omega_0 - \epsilon\theta_E$ is a Kähler metric on \tilde{X} , and $\psi_{\pm, \epsilon} \in C^\infty(\tilde{X})$ converges to ψ_\pm in $L^p(\tilde{X}) \cap L^\infty(\tilde{X} \setminus Z)$ for some $p > 1$. Using (5) and cleverly modifying the C^2 -estimate of Aubin-Yau-Siu, Păun [16] proved the Laplacian estimate for the solution ψ_ϵ away from Z . More precisely, for any compact set $K \subset \tilde{X} \setminus Z$, there exists a constant $A = A(\|\psi\|_\infty, K)$, such that

$$\Delta_\eta \psi_\epsilon \leq A(\|\psi\|_\infty, K) e^{-\psi_-}.$$

From this estimate, we know that the right-hand side of (6) is uniformly $C^{1, \alpha}$ on $\tilde{X} \setminus Z$. By Evans-Krylov's theory ([3]), we know that ψ_ϵ is uniformly $C^{2, \alpha}$ and hence by bootstrapping, $C^{k, \alpha}$ on $\tilde{X} \setminus Z$. Now because ψ_ϵ converges to ψ in C^k norm uniformly away from Z , we get that ψ is smooth on $\tilde{X} \setminus Z$.

One can also prove the regularity on X^{reg} with the help of Kähler-Ricci flow. Starting from the work in [6], this idea has been used several times in the literature to prove the regularity of weak solutions to complex Monge-Ampère equations. Recall that the Kähler-Ricci flow is a solution to the following equation:

$$\begin{cases} \frac{\partial \omega_t}{\partial t} = -\text{Ric}(\omega_t) + \omega_t; \\ \omega(0) = \omega_{\phi_0}. \end{cases} \quad (7)$$

As in the elliptic case, this equation can be transformed into the following Monge-Ampère flow

$$\begin{cases} \frac{\partial \phi}{\partial t} = \log \frac{(\omega_0 + \sqrt{-1}\partial\bar{\partial}\phi)^n}{\Omega} + \phi; \\ \phi(0, \cdot) = \phi_0. \end{cases} \quad (8)$$

To define a solution to this Monge-Ampère flow on the singular variety X , Song-Tian [19] pulled up the flow equation in (8) to \tilde{X} to get:

$$\begin{cases} \frac{\partial \tilde{\phi}}{\partial t} = \log \frac{(\pi^*\omega_0 + \sqrt{-1}\partial\bar{\partial}\tilde{\phi})^n}{\pi^*\Omega} + \tilde{\phi}; \\ \tilde{\phi}(0, \cdot) = \pi^*\phi_0. \end{cases} \quad (9)$$

Theorem 2 ([19]). *Let $\phi_0 \in PSH_p(X, \omega_0)$ for some $p > 1$. Then the Monge-Ampère flow (9) on $\tilde{X} \setminus E$ has a unique solution $\tilde{\phi} \in C^\infty((0, T_0) \times \tilde{X} \setminus E) \cap C^0([0, T_0] \times \tilde{X} \setminus E)$ such that for all $t \in [0, T_0)$, $\tilde{\phi}(t, \cdot) \in L^\infty(\tilde{X}) \cap PSH(\tilde{X}, \pi^*\omega_0)$.*

Since $\tilde{\phi}$ is constant along (connected) fibre of π , $\tilde{\phi}$ descends to a solution $\phi \in C^\infty((0, T_0) \times X^{\text{reg}}) \cap C^0([0, T_0] \times X^{\text{reg}})$ of the Monge-Ampère flow.

Now suppose $\omega_{\text{KE}}^w = \omega_0 + \sqrt{-1}\partial\bar{\partial}\phi_{\text{KE}}^w$ is a weak solution to the equation (2). If one can prove that the solution $\phi(t)$ to (8) with the initial condition $\phi(0) = \phi_{\text{KE}}^w$ is stationary, then it follows

from Theorem (2) that ω_{KE}^w is smooth on X_∞^{reg} . The idea to prove stationarity in [6] is to show that the energy functional is decreasing along the flow solution $\phi(t)$ and to use the uniqueness of weak Kähler-Einstein metrics. These are indeed true in the current case by the work of [1].

To prove Theorem 1, the main observation is that the above arguments can be used to prove the regularity of ω_{KE}^w on X^{orb} as long as one can find a partial resolution by orbifolds: $\pi^{\text{par}} : X^{\text{par}} \rightarrow X$. Indeed, by the next section, there exist orbifold (partial) resolutions. If $\pi^{\text{par}} : X^{\text{par}} \rightarrow X$ is an orbifold resolution, then we can write:

$$K_{X^{\text{par}}} = (\pi^{\text{par}})^* K_X + \sum_i^r a_i E_i - \sum_{j=1}^s b_j F_j,$$

where $E = \cup_{i=1}^r E_i \cup \cup_{j=1}^s F_j$ is now a simple normal crossing divisor within orbifold category (in the sense of Satake [17, 18]). The klt property of X again implies $a_i > 0$ and $0 < b_i < 1$. Then the similar argument as in the proof of regularity of ω_{KE}^w on X_∞^{reg} carries over to the orbifold setting to prove the orbifold regularity of ω_{KE}^w on X^{orb} .

Note that it was already observed in [19, Section 4.3] that if X has only orbifold singularities, then the Kähler-Ricci flow smooths out initial metric to become genuine smooth *orbifold* metric immediately when $t > 0$.

3 Orbifold partial resolution

The results in this section were communicated to us by Chenyang Xu.

Lemma 1 (Resolution of Deligne-Mumford stacks). *Let \mathcal{X} be an integral Deligne-Mumford stack which is of finite type over \mathbb{C} . Then there exists a birational proper representable morphism $g^{\text{sm}} : \mathcal{X}^{\text{sm}} \rightarrow \mathcal{X}$ from a smooth Deligne-Mumford stack \mathcal{X}^{sm} . Furthermore, we can assume that g^{sm} is isomorphic over the smooth locus of \mathcal{X} , and the exceptional locus of g^{sm} is a normal crossing divisorial closed substacks of \mathcal{X}^{sm} .*

Proof. This follows from the functoriality property of resolution of singularities (see [26], [12], [2], [24]). \square

Lemma 2 (Blow up the indeterminacy locus). *Let X be a projective scheme. Let \mathcal{X} be a normal Deligne-Mumford stack with a dense open set $f_U : U \subset \mathcal{X}$, such that U admits a morphism $U \rightarrow X$. Then we can blow up an ideal $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ to obtain a Deligne-Mumford stack $\tilde{\mathcal{X}}$ such that $\tilde{\mathcal{X}} \rightarrow \mathcal{X}$ is isomorphic over U and f_U extends to a morphism $f : \tilde{\mathcal{X}} \rightarrow X$.*

Proof. We can replace X by \mathbb{P}^N . Let $D \subset U$ be the pull back of a hyperplane section H which does not vanish along U , and we let $\mathcal{I} \subset \mathcal{O}_{\mathcal{X}}$ be the ideal of the closure of D in \mathcal{X} . Then the rest of the proof follows from the cases for schemes as in [9, II.7.17.3]. \square

Theorem 3. *Let X be a quasi-projective normal variety. Let X^{orb} be the locus where X only has orbifold singularity. Then there exists $f^{\text{par}} : X^{\text{par}} \rightarrow X$ a proper birational morphism, such that X^{par} only has quotient singularity and f^{par} is an isomorphism over X^{orb} .*

Proof. After taking the closure of $X \subset \mathbb{P}^N$, we can assume X is projective.

By [25, 2.8], we know there is a smooth Deligne-Mumford stack \mathcal{X}^0 whose coarse moduli space is X^{orb} . It follows from [14, Theorem 4.4] that $\mathcal{X}^0 = [Z/G]$ for some quasi-projective scheme Z and linear algebraic group G . Actually, Z can be taken as the frame bundle of X^{orb} and $G = GL_n(\mathbb{C})$. Then by [14, Theorem 5.3], there is a proper Deligne-Mumford stack \mathcal{X} , such that $\mathcal{X}^0 \subset \mathcal{X}$ is a dense open set.

Consider the rational map $f : \mathcal{X} \dashrightarrow X$, by Lemma 2 we know that there is a blow up $\mathcal{Y} \rightarrow \mathcal{X}$ along the indeterminacy locus of f , such that there is a morphism $g : \mathcal{Y} \rightarrow X$. Moreover, by the construction, we know over X^{orb} ,

$$\mathcal{Y}^0 := g^{-1}(X^{\text{orb}}) \cong \mathcal{X}^0.$$

By Lemma 1, we know that there is a smooth Deligne-Mumford stack $h : \mathcal{Y}^{\text{sm}} \rightarrow \mathcal{Y}$, where h is a representable proper birational morphism which is isomorphic over the smooth locus of \mathcal{Y} . In particular, h is isomorphic over \mathcal{Y}^0 .

As \mathcal{X} has finite stabilizer and $\mathcal{Y}^{\text{sm}} \rightarrow \mathcal{Y} \rightarrow \mathcal{X}$ is proper, we know that \mathcal{Y}^{sm} has also finite stabilizer. Thus it follows from [11] that \mathcal{Y}^{sm} admits a coarse moduli space, which we denote by X^{par} . It has a morphism $f^{\text{par}} : X^{\text{par}} \rightarrow X$ by the universal property. And we easily check that they satisfy all the properties. □

Acknowledgement: We would like to thank Chenyang Xu for communicating the results in the last section to us. The first author would like to thank Professor J. Starr for discussions about stacks.

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